ON THE ANNIHILATOR IDEAL OF A HIGHEST WEIGHT VECTOR

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Abstract. In this paper we study the annihilator ideal $ann(v)$ of the highest weight vector $v \in V_\lambda$ where V_λ is an arbitrary finite dimensional irreducible $\mathrm{SL}(E)$-module. We prove there is a decomposition

$$\mathrm{U}_l(\mathfrak{sl}(E)) = \mathrm{U}_l(\mathfrak{n}(\underline{n})) \oplus ann_l(v)$$

where $\mathfrak{n}(\underline{n}) \subseteq \mathfrak{sl}(E)$ is a sub Lie algebra defined in terms of a flag $E_\bullet(\underline{n})$ in E. The decomposition is valid in the case where $1 \leq l \leq m(\lambda)$ where $m(\lambda)$ is a function of the highest weight λ for V_λ. We use this result to study the canonical filtration $\mathrm{U}_l(\mathfrak{g})v \subseteq V_\lambda$ determined by the highest weight vector $v \in V_\lambda$. We give a natural basis for $\mathrm{U}_l(\mathfrak{g})v$ and calculate its dimension. The basis we define is semi canonical in the following sense: It depends on a choice of a basis for the flag $E_\bullet(\underline{n})$ in E

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1. Introduction

In this paper we study the annihilator ideal $ann(v)$ of the highest weight vector $v \in V_\lambda$ where V_λ is an arbitrary finite dimensional irreducible $\mathrm{SL}(E)$-module. We prove there is a decomposition

$$\mathrm{U}_l(\mathfrak{sl}(E)) = \mathrm{U}_l(\mathfrak{n}(\underline{n})) \oplus ann_l(v) \tag{1.0.1}$$

where $\mathfrak{n}(\underline{n}) \subseteq \mathfrak{sl}(E)$ is a sub Lie algebra and $\mathrm{U}_l(\mathfrak{sl}(E))$ is the l'th piece of the canonical filtration of $\mathrm{U}(\mathfrak{sl}(E))$. The Lie algebra $\mathfrak{n}(\underline{n})$ is determined by a flag $E_\bullet(\underline{n})$ in E defined in terms of the highest weight λ for V_λ. The decomposition 1.0.1 is valid in the case when $1 \leq l \leq m(\lambda)$ where $m(\lambda)$ is a function of the highest weight λ of V_λ. We use the decomposition 1.0.1 to study the canonical filtration $\mathrm{U}_l(\mathfrak{sl}(E))v$ in V_λ determined by the highest weight vector v in V_λ. We give a natural basis

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$\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{sl}(E))v$ and calculate its dimension. The basis we define is semi canonical in the following sense: It depends on the choice of basis B for $E_\bullet(\underline{n})$.

The canonical filtration

$$\mathrm{U}_1(\mathfrak{sl}(E))v \subseteq \cdots \subseteq \mathrm{U}_l(\mathfrak{sl}(E))v \subseteq V_\lambda \tag{1.0.2}$$

of an irreducible $\mathrm{SL}(E)$-module V_λ is a filtration of P-modules where P in $\mathrm{SL}(E)$ is the parabolic subgroup stabilizing the highest weight vector v in V_λ. The irreducible $\mathrm{SL}(E)$-module V_λ may by the Borel-Weil-Bott formula be constructed geometrically as the $\mathrm{SL}(E)$-module of global sections of an invertible sheaf on the flag variety $\mathrm{SL}(E)/P$. There is an isomorphism

$$V_\lambda \cong \mathrm{H}^0(\mathrm{SL}(E)/P, \mathcal{L}(\underline{l}))$$

of $\mathrm{SL}(E)$-modules where $\mathcal{L}(\underline{l})$ is in $\mathrm{Pic}^{\mathrm{SL}(E)}(\mathrm{SL}(E)/P)$. It is well known the cohomology group $\mathrm{H}^0(\mathrm{SL}(E)/P, \mathcal{L}(\underline{l}))$ has a basis given in terms of standard monomials. There is a generalized Plücker embedding

$$i : \mathrm{SL}(E)/P \to \mathbb{P}(\wedge^{n_1} E^*) \times \cdots \times \mathbb{P}(\wedge^{n_k} E^*)$$

and a standard monomial basis for $\mathrm{H}^0(\mathrm{SL}(E)/P, \mathcal{L}(\underline{l}))$ is expressed in terms of monomials in homogeneous coordinates on the projective spaces $\mathbb{P}(\wedge^{n_i} E^*)$. The basis $\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{sl}(E))v$ constructed in this paper is defined in terms of the Lie algebra $\mathfrak{n}(\underline{n})$ and its l'th piece $\mathrm{U}_l(\mathfrak{n}(\underline{n}))$ of the canonical filtration of the enveloping algebra $\mathrm{U}(\mathfrak{n}(\underline{n}))$. A basis B for E compatible with the flag $E_\bullet(\underline{n})$ of determined by λ gives in a canonical way rise to a basis $B(\underline{n})$ for $\mathfrak{n}(\underline{n})$. The basis $B(\underline{n})$ gives in a canonical way the basis $\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{sl}(E))v$. It is an unsolved problem to describe the basis $\mathcal{B}(l, \underline{n}, B)$ in terms of the standard monomial basis for $\mathrm{H}^0(\mathrm{SL}(E)/P, \mathcal{L}(\underline{l}))$ induced by the Plücker embedding.

The paper is organized as follows: In section two we use the explicit construction from the Appendix and general properties of the universal enveloping algebra and the annihilator ideal to calculate the decomposition 1.0.1 for the annihilator ideal $ann(v)$ of any highest weight vector v in V_λ where V_λ is an arbitrary finite dimensional irreducible $\mathfrak{sl}(E)$-module. This is Theorem 2.20. We also give a natural basis $\mathcal{B}(l, \underline{n}, B)$ for the l'th piece of the canonical filtration $\mathrm{U}_l(\mathfrak{sl}(E))v$ and calculate its dimension. This is Corollary 2.21.

In section three we give an elementary construction of all finite dimensional irreducible $\mathrm{SL}(E)$-modules and their highest weight vectors using multilinear algebra (see Theorem 3.2). This construction is needed in section two for calculational purposes.

2. On the annihilator ideal of a highest weight vector

In this section we study the annihilator ideal $ann(v)$ of the highest weight vector v in an arbitrary finite dimensional irreducible $\mathrm{SL}(E)$-module V_λ. We use properties of $ann(v)$ to study the canonical filtration $\mathrm{U}_l(\mathfrak{sl}(E))v \subseteq V_\lambda$ defined by the highest weight vector v in V_λ. We give a natural basis for $\mathrm{U}_l(\mathfrak{sl}(E))v$ and calculate its dimension as function of l.

Using the explicit construction of V_λ and v given in the Appendix we calculate the parabolic subgroup P in $\mathrm{SL}(E)$ stabilizing the line L_v spanned by v. We use this to calculate a sub Lie algebra $\mathfrak{n}(\underline{n})$ in $\mathfrak{sl}(E)$ with the following property: There

is a decomposition

$$\mathrm{U}_l(\mathfrak{sl}(E)) \cong \mathrm{U}_l(\mathfrak{n}(\underline{n})) \oplus ann_l(v). \tag{2.0.3}$$

where $1 \leq l \leq m(\lambda)$. The function $m(\lambda)$ is a function of the highest weight λ for V_λ. This result is the main result of this section (see Theorem 2.20). We use this to give a natural basis $\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{sl}(E))v$ and to calculate $dim(\mathrm{U}_l(\mathfrak{sl}(E))v)$ as a function of l (see Corollary 2.21). The basis $\mathcal{B}(l, \underline{n}, B)$ is semi canonical in the following sense: A choice of basis B for E compatible with the flag $E_\bullet(\underline{n})$ determines the basis $\mathcal{B}(l, \underline{n}, B)$.

Notation: Let K be a fixed algebraically closed field of characteristic zero and let E be an n-dimensional K-vector space with basis $B = \{e_1, .., e_n\}$. Let $G = \mathrm{SL}(E)$ and $\mathfrak{g} = \mathfrak{sl}(E)$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ of diagonal matrices with trace zero. It follows the pair $(\mathfrak{g}, \mathfrak{h})$ is a split semi simple Lie algebra in the sense of [3]. Let the roots $R(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ be denoted R. Let $\tilde{B}$ be a basis for R and let R_+ and R_- be the negative and positive roots corresponding to $\tilde{B}$. This choice determines by [3] a decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

of $\mathfrak{g}$ called the *triangular decomposition* defined by $\tilde{B}$.

It follows $\mathfrak{n}_-$ is the sub algebra of $\mathfrak{g}$ of strictly lower triangular matrices and $\mathfrak{n}_+$ the sub algebra of strictly upper triangular matrices of $\mathfrak{g}$. Let $\mathfrak{h}^*$ be defined as follows: If $x \in \mathfrak{h}$ is the following element:

$$x = \begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix}$$

let $L_i(x) = a_i$. It follows

$$\mathfrak{h}^* = K\{L_1, .., L_n\}/L_1 + \cdots + L_n.$$

Definition 2.1. Let $\omega_i = L_1 + \cdots + L_i$ for $1 \leq i \leq n-1$ be the *fundamental weights* for $\mathfrak{g}$.

Let V_λ be an irreducible finite dimensional $\mathfrak{g}$-module with highest weight vector v and highest weight λ. It follows from the Appendix λ is as follows:

$$\lambda = \sum_{i=1}^{k} l_i \omega_{n_i}$$

where $l_i \geq 1$ and

$$1 \leq n_1 < \cdots < n_k \leq n-1$$

are integers. Recall from the Appendix, Theorem 3.2 we get an explicit description of V_λ:

$$V_\lambda \cong \mathrm{U}(\mathfrak{g})v \subseteq W(\underline{l}, \underline{n})$$

where

$$v = w_1^{l_1} \otimes \cdots \otimes w_k^{l_k} \in W(\underline{l}, \underline{n})$$

is the highest weight vector and $\mathrm{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Let L_v be the line spanned by v and let P in G be the subgroup fixing L_v.

Recall from the Appendix, 3.0.3 the following: Let the vector space E_i have basis $B_i = \{e_1, .., e_{n_i}\}$ for $1 \leq i \leq k$. We get a flag

$$E_\bullet(\underline{n}) : 0 \neq E_1 \subseteq \cdots \subseteq E_k \subseteq E_{k+1} = E$$

of type $\underline{n}$ in E.

Let $P(\underline{n})$ in $\mathrm{SL}(E)$ be the subgroup fixing the flag $E_\bullet(\underline{n})$.

Lemma 2.2. *There is an equality $P = P(\underline{n})$ of subgroups of* $\mathrm{SL}(E)$.

Proof. The proof is an exercise. □

Let $\mathfrak{p}(\underline{n}) = Lie(P(\underline{n}))$. Since L_v is $P(\underline{n})$-stable we get a character

$$\rho_v : \mathfrak{p}(\underline{n}) \to \mathrm{End}(L_v) \cong K.$$

The Lie algebra $\mathfrak{p}(\underline{n})$ looks as follows: An element $x \in \mathfrak{p}(\underline{n})$ is on the following form:

$$x = \begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ 0 & 0 & \cdots & \cdots & * \\ 0 & 0 & 0 & \vdots & A_{k+1} \end{pmatrix}$$

where A_i is a $d_i \times d_i$-matrix with coefficients in K, and $tr(x) = 0$.

Let E^* have basis $e_1^*, .., e_n^*$ and let E_i^* have basis $e_1^*, .., e_{n_i}^*$. It follows there is a canonical isomorphism

$$E \otimes E^* \cong \mathrm{End}_K(E)$$

defined by

$$(v \otimes f)(e) = f(e)v.$$

This isomorphism induce injections

$$E_i \otimes E_i^* \to \mathrm{End}(E) \cong E \otimes E^*$$

since any element $e_k \otimes e_l^* \in E_i \otimes E_i^*$ is an element in $E \otimes E^*$. Define the following:

$$B^l = \{e_i \otimes e_j^* : n_{l-1} < i \leq n_l, n_l < j \leq n\}$$

for $1 \leq l \leq k$. Let

$$B(\underline{n}) = \{B^1, B^2, .., B^k\}. \tag{2.2.1}$$

Let $\mathfrak{n}(\underline{n}) \subseteq \mathfrak{sl}(E)$ be the subspace spanned by the vectors in $B(\underline{n})$.

Lemma 2.3. *The vector space $\mathfrak{n}(\underline{n})$ does not depend on the choice of basis B for E compatible with the flag $E_\bullet(\underline{n})$.*

Proof. Assume we have chosen a basis $C = \{f_1, .., f_n\}$ for E with $C_i = \{f_1, .., f_{n_i}\} = E_i$. Assume $[I]^B_C$ is a basechange from B to C compatible with the flag $E_\bullet(\underline{n})$. It follows $[I]^B_C$ looks as follows:

$$M = [I]^B_C = \begin{pmatrix} I_1 & * & * & \cdots & * \\ 0 & I_2 & * & \cdots & * \\ 0 & 0 & \cdots & \cdots & * \\ 0 & 0 & 0 & \vdots & I_{k+1} \end{pmatrix}$$

where $|M| \neq 0$ and I_i is a $d_i \times d_i$-matrix with coefficients in K. The base change matrix from the dual basis $B^* = \{e_1^*, .., e_n^*\}$ to $C^* = \{f_1^*, .., f_n^*\}$ is the transpose of

M. It follows the vectors $f^*_{n_l+1}, .., f^*_{n_l}$ are included in the vector space spanned by the vectors

$$e^*_{n_l+1}, .., e^*_n.$$

Let C^l be the vectors $f_i \otimes f^*_j$ with $n_{l-1} < i \leq n_l$ and $n_l < j \leq n$. It follows the vector $f_i \otimes f^*_j$ is included in the space spanned by the vectors $e_i \otimes e^*_j$ with $1 \leq i \leq n_l$ and $n_l + 1 \leq j \leq n$. It follows the vectors in C^l lie in the vector space spanned by the set

$$\{B^1, .., B^l\}.$$

It follows the vector space spanned by the set $\{C^1, .., C^k\}$ equals the space spanned by the set $\{B^1, .., B^k\}$. A similar argument proves the space spanned by the set $\{B^1, .., B^k\}$ equals the space spanned by the set $\{C^1, .., C^k\}$ and the Lemma is proved since $\mathfrak{n}(\underline{n})$ is by definition the vector space spanned by the set $\{B^1, .., B^k\}$. □

The following holds:

Lemma 2.4. *The subspace $\mathfrak{n}(\underline{n}) \subseteq \mathfrak{sl}(E)$ is a Lie algebra. Any choice of basis B for E compatible with the flag $E_\bullet(\underline{n})$ determines by the construction in 2.2.1 a basis $B(\underline{n})$ for $\mathfrak{n}(\underline{n})$.*

Proof. One checks $\mathfrak{n}(\underline{n})$ is the elements x in $\mathfrak{g}$ on the following form:

$$x = \begin{pmatrix} A_1 & 0 & \cdots & 0 & 0 \\ * & A_2 & \cdots & 0 & 0 \\ * & * & \cdots & A_k & 0 \\ * & * & \cdots & * & A_{k+1} \end{pmatrix}$$

where A_i is a $d_i \times d_i$-matrix with zero entries. It follows $\mathfrak{n}$ is closed under the Lie bracket. The second claim of the Lemma follows from the construction above and the Lemma is proved. □

It follows we get a direct sum decomposition

$$\mathfrak{sl}(E) \cong \mathfrak{n}(\underline{n}) \oplus \mathfrak{p}(\underline{n}) \tag{2.4.1}$$

which only depends on the choice of flag $E_\bullet(\underline{n})$ in E of type $\underline{n}$. The direct sum Lie algebra $\mathfrak{n}(\underline{n}) \oplus \mathfrak{p}(\underline{n})$ is not isomorphic to $\mathfrak{sl}(E)$ as a Lie algebra.

Definition 2.5. Let the Lie algebra $\mathfrak{n}(\underline{n})$ be the *complementary Lie algebra* of the flag $E_\bullet(\underline{n})$. Let the Lie algebra $\mathfrak{p}(\underline{n})$ be the *stabilizer Lie algebra* of $E_\bullet(\underline{n})$.

Since the Lie algebra $\mathfrak{n}(\underline{n})$ by Lemma 2.3 only depends on the flag $E_\bullet(\underline{n})$ in E it follows Definition 2.5 is well defined.

Recall the character

$$\rho_v : \mathfrak{p}(\underline{n}) \rightarrow \mathrm{End}(L_v).$$

We get by Proposition 3.1

$$\rho_v(x) = \sum_{i=1}^k l_i(tr(A_1) + \cdots + tr(A_i)).$$

Let $\mathfrak{p}_v = Ker(\rho_v) \subseteq \mathfrak{p}(\underline{n})$. There is an equality

$$\mathfrak{p}_v = \{x \in \mathfrak{p}(\underline{n}) : x(v) = 0\}.$$

The Lie algebra $\mathfrak{p}_v$ is the *isotropy Lie algebra* of the line L_v. We get an exact sequence of Lie algebras

$$0 \to \mathfrak{p}_v \to \mathfrak{p}(\underline{n}) \to \mathrm{End}(L_v) \to 0.$$

Since $dim(\mathrm{End}(L_v)) = 1$ it follows there is an element $x \in \mathfrak{p}(\underline{n})$ with $x(v) = \alpha v$, $0 \neq \alpha \in K$ with the following property: There is a direct sum decomposition $\mathfrak{p}(\underline{n}) = \mathfrak{p}_v \oplus L_x$ where L_x is the line spanned by x. We may choose a basis for $\mathfrak{p}(\underline{n})$ on the form

$$\{x, y_1, .., y_E\}$$

with $\rho_v(y_i) = 0$ for $1 \leq i \leq E$.

Pick a basis $\{x_1, .., x_D\}$ for $\mathfrak{n}(\underline{n})$.

Proposition 2.6. *The natural map*

$$f : \mathrm{U}(\mathfrak{n}(\underline{n})) \otimes_K \mathrm{U}(\mathfrak{p}(\underline{n})) \to \mathrm{U}(\mathfrak{g})$$

defined by

$$f(x \otimes y) = xy$$

is an isomorphism of vector spaces.

Proof. For a proof see [3], Proposition 2.2.9. □

Definition 2.7. Define the following for all $l \geq 1$:

$$\mathrm{U}_l(\mathfrak{n}(\underline{n}), \mathfrak{p}(\underline{n})) = \sum_{i+j=l} \mathrm{U}_i(\mathfrak{n}(\underline{n})) \otimes_K \mathrm{U}_j(\mathfrak{p}(\underline{n})) \subseteq \mathrm{U}(\mathfrak{n}(\underline{n})) \otimes_K \mathrm{U}(\mathfrak{p}(\underline{n})).$$

Lemma 2.8. *The isomorphism $f : \mathrm{U}(\mathfrak{n}(\underline{n})) \otimes_K \mathrm{U}(\mathfrak{p}(\underline{n})) \cong \mathrm{U}(\mathfrak{g})$ induce an isomorphism*

$$f : \mathrm{U}_l(\mathfrak{n}(\underline{n}), \mathfrak{p}(\underline{n})) \cong \mathrm{U}_l(\mathfrak{g})$$

of vector spaces.

Proof. Let $x \otimes y \in \mathrm{U}_i(\mathfrak{n}(\underline{n})) \otimes \mathrm{U}_j(\mathfrak{p}(\underline{n}))$ with $i+j = l$. It follows $f(x \otimes y) = xy \in \mathrm{U}_l(\mathfrak{g})$ hence $f(\mathrm{U}_l(\mathfrak{n}(\underline{n}), \mathfrak{p}(\underline{n})) \subseteq \mathrm{U}_l(\mathfrak{g})$. Assume

$$\eta = x_1^{v_1} \cdots x_D^{v_D} x^q y_1^{u_1} \cdots u_E^{u_E} \in \mathrm{U}_l(\mathfrak{g}).$$

Let $i = \sum v_n$ and $j = \sum u_m + q$. It follows $i+j = l$ hence the element $z = x_1^{v_1} \cdots x_D^{v_D}$ is in $\mathrm{U}_i(\mathfrak{n}(\underline{n}))$ and $y = x^q y_1^{u_1} \cdots u_E^{u_E}$ is in $\mathrm{U}_j(\mathfrak{p}(\underline{n}))$. It follows $z \otimes y \in \mathrm{U}_l(\mathfrak{n}(\underline{n}), \mathfrak{p}(\underline{n}))$ and $f(z \otimes y) = zy \in \mathrm{U}_l(\mathfrak{g})$ and the Lemma is proved. □

Let $\mathbf{1}_\mathfrak{p} \in \mathrm{U}(\mathfrak{p}(\underline{n}))$ be the multiplicative identity.

Definition 2.9. Let $l \geq 1$ be an integer. Define the following:

$$\mathrm{U}_l(\mathfrak{n}(\underline{n})) \otimes \mathbf{1}_\mathfrak{p} = \{x \otimes \mathbf{1}_\mathfrak{p} : x \in \mathrm{U}_l(\mathfrak{n}(\underline{n}))\}.$$

$$W_l = \{x \otimes w(y - \rho_v(y)\mathbf{1}_\mathfrak{p}) : x \in \mathrm{U}_i(\mathfrak{n}(\underline{n})), y \in \mathfrak{p}(\underline{n}), w(y - \rho_v(y)\mathbf{1}_\mathfrak{p}) \in \mathrm{U}_j(\mathfrak{p}(\underline{n})), i+j = l\}.$$

Lemma 2.10. *The natural map*

$$\phi : \mathrm{U}_l(\mathfrak{n}(\underline{n})) \otimes \mathbf{1}_\mathfrak{p} \oplus W_l \to \mathrm{U}_l(\mathfrak{n}(\underline{n}), \mathfrak{p}(\underline{n}))$$

defined by

$$\phi(x, y) = x + y$$

is an isomorphism of vector spaces.

Proof. We first prove $\mathrm{U}_l(\mathfrak{n}(\underline{n}))\otimes \mathbf{1}_{\mathfrak{p}}\cap W_l=\{0\}$ as subspaces of $\mathrm{U}_l(\mathfrak{n}(\underline{n}),\mathfrak{p}(\underline{n}))$ for all $l\geq 1$: Let

$$\omega = x\otimes w(y-\rho_v(y)\mathbf{1}_{\mathfrak{p}})\in W_l.$$

One of the following holds:

$$\omega = x\otimes wy_i \tag{2.10.1}$$

with $y_i\in \mathfrak{p}_v$.

$$\omega = x\otimes w(x-\alpha(x)\mathbf{1}_{\mathfrak{p}}) \tag{2.10.2}$$

with $\alpha(x)\neq 0$.

$$\omega = 0. \tag{2.10.3}$$

If $\omega\in \mathrm{U}_l(\mathfrak{n}(\underline{n}))\otimes \mathbf{1}_{\mathfrak{p}}$ it follows $\omega=0$ and the claim follows. Assmue $\phi(x,y)=x+y=0$. It follows $y=-x\in W_l$ hence $-x=0=x=y$. Hence ϕ is an injective map. We next prove ϕ is surjective: Write

$$\mathrm{U}_l(\mathfrak{n}(\underline{n}),\mathfrak{p}(\underline{n}))=\mathrm{U}_l(\mathfrak{n}(\underline{n}))\otimes \mathbf{1}_{\mathfrak{p}}+\sum_{i=1}^{l}\mathrm{U}_{l-i}(\mathfrak{n}(\underline{n}))\otimes \mathrm{U}_i(\mathfrak{p}(\underline{n})).$$

If $x\otimes\mathbf{1}_{\mathfrak{p}}\in \mathrm{U}_l(\mathfrak{n}(\underline{n}))\otimes\mathbf{1}_{\mathfrak{p}}$ it follows $(x\otimes\mathbf{1}_{\mathfrak{p}},0)\in \mathrm{U}_l(\mathfrak{n}(\underline{n}))\otimes\mathbf{1}_{\mathfrak{p}}\oplus W_l$ and $\phi(x\otimes\mathbf{1}_{\mathfrak{p}},0)=x\otimes\mathbf{1}_{\mathfrak{p}}$. Assume

$$\omega\in \mathrm{U}_{l-i}(\mathfrak{n}(\underline{n}))\otimes \mathrm{U}_i(\mathfrak{p}(\underline{n}))-\mathrm{U}_{l-(i-1)}(\mathfrak{n}(\underline{n}))\otimes \mathrm{U}_{i-1}(\mathfrak{p}(\underline{n})).$$

It follows

$$\omega = x_1^{v_1}\cdots x_D^{v_D}\otimes x^q y_1^{u_1}\cdots y_E^{u_E}$$

with

$$\sum v_j = l-i$$

and

$$\sum u_j+q=i.$$

Let k be minimal with $u_k\geq 1$. It follows

$$\omega = x_1^{v_1}\cdots x_D^{v_D}\otimes x^q y_1^{u_1}\cdots y_k^{u_k}=$$

$$x_1^{v_1}\cdots x_D^{v_D}\otimes x^q y_1^{u_1}\cdots y_k^{u_k-1}(y_k-\rho_v(y_k)\mathbf{1}_{\mathfrak{p}})\in W_l.$$

It follows

$$\phi(0,\omega)=\omega.$$

Assume

$$\omega = x_1^{v_1}\cdots x_D^{v_D}\otimes x^q$$

aith $q\geq 1$. We may write

$$x^q=\alpha(x)^q\mathbf{1}_{\mathfrak{p}}+y(x-\alpha(x)\mathbf{1}_{\mathfrak{p}})$$

for some y. It follows

$$\omega = x_1^{v_1}\cdots x_D^{v_D}\otimes x^q\alpha(x)^q\otimes\mathbf{1}_{\mathfrak{p}}+x_1^{v_1}\cdots x_D^{v_D}\otimes x^q\otimes y(x-\alpha(x)\mathbf{1}_{\mathfrak{p}}).$$

We get

$$\phi(x_1^{v_1}\cdots x_D^{v_D}\otimes x^q\alpha(x)^q\otimes\mathbf{1}_{\mathfrak{p}},x_1^{v_1}\cdots x_D^{v_D}\otimes x^q\otimes y(x-\alpha(x)\mathbf{1}_{\mathfrak{p}}))=\omega.$$

It follows ϕ is surjective, and the Lemma is proved. □

Let $\mathbf{1}_{\mathfrak{g}}\in \mathrm{U}(\mathfrak{g})$ be the multiplicative identity.

Definition 2.11. Let

$$char(\rho_v) = \{x(y - \rho_v(y)\mathbf{1}_{\mathfrak{g}}) : x \in \mathrm{U}(\mathfrak{g}), y \in \mathfrak{p}(\underline{n})\} \subseteq \mathrm{U}(\mathfrak{g})$$

be the *character ideal* associated to the highest weight vector $v \in V_\lambda$. Let $char_l(\rho_v) = char(\rho_v) \cap \mathrm{U}_l(\mathfrak{g})$ be its canonical filtration.

Since the left ideal $char(\rho_v) \subseteq \mathrm{U}(\mathfrak{g})$ depends on the line $L_v \subseteq V_\lambda$ which is canonical, it follows $char(\rho_v)$ is well defined.

The embedding of Lie algebras $\mathfrak{n}(\underline{n}) \subseteq \mathfrak{g}$ induce a canonical embedding of associative rings $\mathrm{U}(\mathfrak{n}(\underline{n})) \subseteq \mathrm{U}(\mathfrak{g})$ and canonical embeddings of filtrations $\mathrm{U}_l(\mathfrak{n}(\underline{n})) \subseteq \mathrm{U}_l(\mathfrak{g})$ for all $l \geq 1$. We have inclusions

$$\mathrm{U}_l(\mathfrak{n}(\underline{n})) \otimes \mathbf{1}_{\mathfrak{p}}, W_l \subseteq \mathrm{U}_l(\mathfrak{n}(\underline{n}), \mathfrak{p}(\underline{n}))$$

of vector spaces for all $l \geq 1$.

Lemma 2.12. *The map* $f : \mathrm{U}(\mathfrak{n}(\underline{n})) \otimes_K \mathrm{U}(\mathfrak{p}(\underline{n})) \to \mathrm{U}(\mathfrak{g})$ *induce isomorphisms*

$$f : \mathrm{U}_l(\mathfrak{n}(\underline{n})) \otimes \mathbf{1}_{\mathfrak{p}} \cong \mathrm{U}_l(\mathfrak{n}(\underline{n})) \tag{2.12.1}$$

$$f : W_l \cong char_l(\rho). \tag{2.12.2}$$

of vector spaces.

Proof. We prove 2.12.1: It is clear $f(\mathrm{U}_l(\mathfrak{n}(\underline{n})) \otimes \mathbf{1}_{\mathfrak{p}}) \subseteq \mathrm{U}_l(\mathfrak{n}(\underline{n}))$. Assume $x \in \mathrm{U}_l(\mathfrak{n}(\underline{n}))$ it follows $x \otimes \mathbf{1}_{\mathfrak{p}} \in \mathrm{U}_l(\mathfrak{n}(\underline{n})) \otimes \mathbf{1}_{\mathfrak{p}}$ and $f(x \otimes \mathbf{1}_{\mathfrak{p}}) = x$ hence claim 2.12.1 is true. We prove 2.12.2: It is clear $f(W_l) \subseteq char_l(\rho_v)$. Assume

$$\omega \in char_l(\rho_v) = char(\rho_v) \cap \mathrm{U}_l(\mathfrak{g})$$

is a monomial. It follows

$$\omega = x_1^{v_1} \cdots x_D^{v_d} x^q y_1^{u_1} \cdots y_E^{u_E}(y - \rho_v(y)\mathbf{1}_{\mathfrak{p}})$$

with $y \in \mathfrak{p}(\underline{n})$. Let $\sum v_n = i$ and $\sum u_m + q + 1 = j$. It follows $i + j \leq l$. We get

$$\eta = x_1^{v_1} \cdots x_D^{v_d} \otimes x^q y_1^{u_1} \cdots y_E^{u_E}(y - \rho_v(y)\mathbf{1}_{\mathfrak{p}}) \in W_l$$

and $f(\eta) = \omega$. Hence claim 2.12.2 follows and the Lemma is proved. □

There is for all $l \geq 1$ a map

$$\phi_l : \mathrm{U}_l(\mathfrak{n}(\underline{n})) \oplus char_l(\rho_v) \to \mathrm{U}_l(\mathfrak{g})$$

of vector spaces defined by

$$\phi_l(x, y) = x + y.$$

The following holds:

Theorem 2.13. *The map* ϕ_l *defines for all* $l \geq 1$ *an isomorphism*

$$\phi_l : \mathrm{U}_l(\mathfrak{n}(\underline{n})) \oplus char_l(\rho_v) \cong \mathrm{U}_l(\mathfrak{g})$$

of vector spaces.

Proof. The Theorem follows from Lemma 2.10 and Lemma 2.12. □

Note: Theorem 2.13 is valid over an arbitrary field.

Definition 2.14. Let

$$ann(v)) = \{x \in \mathrm{U}(\mathfrak{g}) : x(v) = 0\}$$

be the *annihilator ideal* of $v \in V_\lambda$. Let $ann_l(v) = ann(v) \cap \mathrm{U}_l(\mathfrak{g})$ be its canonical filtration.

The annihilator ideal $ann(v)$ is uniquely determined by the line $L_v \subseteq V_\lambda$ which is canonical, hence $ann(v)$ is well defined. There is an inclusion of left ideals

$$char(\rho_v) \subseteq ann(v)$$

and an inclusion

$$char_l(\rho_v) \subseteq ann_l(v)$$

of filtrations for all $l \geq 1$.

Note: Assume K is algebraically closed. By [3] Section 7, remark 7.8.25 it follows every primitive ideal I of $\mathrm{U}(\mathfrak{g})$ is on the form $ann(v)$ for some highest weight vector v in a finite dimensional irreducible G-module V_λ.

There is an exact sequence

$$0 \to ann(v) \otimes_K L_v \to \mathrm{U}(\mathfrak{g}) \otimes_K L_v \to V_\lambda \to 0$$

of G-modules and an exact sequence

$$0 \to ann_l(v) \otimes_K L_v \to \mathrm{U}_l(\mathfrak{g}) \otimes_K L_v \to \mathrm{U}_l(\mathfrak{g})v \to 0$$

of P-modules. Here $\mathrm{U}_l(\mathfrak{g})v$ in V_λ is the P-module spanned by $\mathrm{U}_l(\mathfrak{g})$ and v.

Definition 2.15. Let $\{\mathrm{U}_l(\mathfrak{g})v\}_{l\geq 1}$ be the *canonical filtration* of V_λ.

Since the terms $\mathrm{U}_l(\mathfrak{g})v$ are uniquely determined by the line L_v which is a canonical line in V_λ it follows we get a canonical filtration

$$\mathrm{U}_1(\mathfrak{g})v \subseteq \cdots \subseteq \mathrm{U}_l(\mathfrak{g})v \subseteq V_\lambda$$

of V_λ by P-modules.

Example 2.16. *Representations of semi simple algebraic groups.*

Let G be a semi simple linear algebraic group over K and let V_λ be a finite dimensional irreducible G-module with highest weight vector $v \in V_\lambda$. Let $P_v \subseteq G$ be the parabolic subgroup fixing the line L_v spanned by v. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p}_v = Lie(P_v)$. We get a canonical filtration

$$\mathrm{U}_1(\mathfrak{g})v \subseteq \cdots \subseteq \mathrm{U}_l(\mathfrak{g})v \subseteq V_\lambda \tag{2.16.1}$$

of V_λ by P_v-modules. Hence Definition 2.15 makes sense for any finite dimensional irreducible representation V_λ of any semi simple linear algebraic group G.

Example 2.17. *On generators for the annihilator ideal $ann(v)$.*

Given an irreducible finite dimensional $\mathrm{SL}(E)$-module V_λ with highest weight vector v and highest weight $\lambda \in \mathfrak{h}^*$ it follows by the results of [3] section 7, generators of the annihilator ideal $ann(v)$ are completely described. A set of generators for the ideal $ann(v) \subseteq \mathrm{U}(\mathfrak{g})$ is given as a function of the highest weight λ. In the following we use the notation from [3], Chapter 7. Recall we have chosen a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. Let R be the *root system* for $\mathfrak{g}$ and let $\tilde{B}$ be a *basis* for R. We let $\tilde{B}$ be as follows:

$$\tilde{B} = \{L_1 - L_2, L_2 - L_3, .., L_{n-1} - L_n\}.$$

Let R_- be the *negative roots* and let R_+ be the *positive roots*. Let P_{++} be the *fundamental weights*. Let $\delta = \frac{1}{2} = \sum_{\alpha \in R_+} \alpha$, $\mathfrak{n}_+ = \sum_{\alpha \in R_+} \mathfrak{g}^\alpha$ and $\mathfrak{n}_- = \sum_{\alpha \in R_-} \mathfrak{g}^\alpha$. Let $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}_+$ and $\mathfrak{b}_- = \mathfrak{h} + \mathfrak{n}_-$.

Let V_λ be the irreducible finite dimensional $\mathfrak{g}$-module with highest weight

$$\lambda = \sum_{i=1}^{k} l_i \omega_{n_i}$$

where

$$1 \leq n_1 < n_2 < \cdots < n_k \leq n = n_k$$

and $l_1, .., l_k \geq 1$. It follows by the results of [3], Chapter 7 V_λ is isomorphic to $L(\lambda+\delta)$ - a quotient of the *Verma module* $M(\lambda+\delta)$ associated to $\lambda+\delta$ by a maximal sub-G-module. By [3], Theorem 7.2.6 this sets up a one to one correspondence between the set P_{++} and the set of finite dimensional irreducible $\mathfrak{g}$-modules.

In [3], Proposition 7.2.7 the annihilator ideal $ann(v)$ is calculated. Let $\beta_i = L_i - L_{i+1}$ for $1 \leq i \leq n-1$. It follows $\mathfrak{g}^{\beta_i} = K(E_{i,i+1})$ and $\mathfrak{g}^{-\beta_i} = K(E_{i+1,i})$. It follows

$$[E_{ii}, E_{jj}] = E_{ii} - E_{jj}.$$

Let $X_{-\beta_i} = E_{i+1,i}$. Let $0 \neq H_{\beta_i} \in [\mathfrak{g}^{\beta_i}, \mathfrak{g}^{-\beta_i}]$. It follows we may choose $H_{\beta_i} = E_{i,i} - E_{i+1,i+1}$. Let

$$m_{\beta_i} = \lambda(H_{\beta_i}) + 1.$$

Lemma 2.18. *The following holds:*

$$m_{\beta_i} = l_j + 1 \text{ if } i = n_j \tag{2.18.1}$$

$$m_{\beta_i} = 1 \text{ if } i \neq n_j. \tag{2.18.2}$$

Proof. By definition $H_{\beta_i} = E_{i,i} - E_{i+1,i+1}$. Also

$$\omega_{n_j} = L_1 + \cdots + L_{n_j}.$$

Assume $i = n_j$. It follows

$$\lambda(H_{\beta_i}) + 1 =$$

$$(l_1\omega_{n_1} + \cdots + l_j\omega_{n_j} + \cdots + l_k\omega_{n_k})(E_{i,i}) -$$

$$(l_1\omega_{n_1} + \cdots + l_j\omega_{n_j} + \cdots + l_k\omega_{n_k})(E_{i+1,i+1}) + 1 =$$

$$l_j + \cdots + l_k - l_{j+1} - \cdots - l_k + 1 = l_j + 1$$

and claim 2.18.1 follows. Claim 2.18.2 is proved in a similar fashion and the Lemma is proved. □

The following Lemma gives a description of the l'th piece $ann_l(v)$ in many cases: Let $m(\lambda) = min_{i=1}^{k}\{l_i\}$.

Lemma 2.19. *For all $1 \leq l \leq m(\lambda)$ there is an equality*

$$ann_l(v) = char_l(\rho_v).$$

Proof. Let $I(v) \subseteq \mathrm{U}(\mathfrak{g})$ be the left ideal defined as follows:

$$I(v) = \mathrm{U}(\mathfrak{g})\mathfrak{n}_+ + \sum_{x \in \mathfrak{h}} \mathrm{U}(\mathfrak{g})(x - \lambda(x)\mathbf{1}_{\mathfrak{g}}).$$

By [3], Proposition 7.2.7 it follows there is an equality

$$ann(v) = I(v) + \sum_{\beta \in B} \mathrm{U}(\mathfrak{n}_-) X_{-\beta}^{m_\beta}.$$

Let $I^l(v) = I(v) \cap \mathrm{U}_l(\mathfrak{g})$. Let

$$J^l(v) = (\sum_{\beta \in B} \mathrm{U}(\mathfrak{n}_-) X_{-\beta}^{m_\beta}) \cap \mathrm{U}_l(\mathfrak{g}).$$

It follows

$$J^l(v) = \sum_{i \neq n_j} \mathrm{U}_{l-1}(\mathfrak{n}_-) X_{-\beta_i} + \sum_{i = n_j} \mathrm{U}_{l-l_j-1}(\mathfrak{n}_-) X_{-\beta_{n_j}}^{l_j+1}.$$

If $1 \leq l \leq m(\lambda)$ it follows

$$J^l(v) = \sum_{i \neq n_j} \mathrm{U}_{l-1}\, X_{-\beta_i}.$$

We get

$$ann_l(v) = I^l(v) + J^l(v).$$

By definition we have

$$char_l(\rho_v) \subseteq ann_l(v)$$

for all $l \geq 1$. There is an inclusion

$$I^l(v) \subseteq char_l(\rho_v)$$

for all $l \geq 1$. When $1 \leq l \leq m(\lambda)$ there is an inclusion

$$J^l(v) \subseteq char_l(\rho_v).$$

It follows

$$ann_l(v) = I^l(v) + J^l(v) \subseteq char_l(\rho_v)$$

and the Lemma follows. □

There is for every $l \geq 1$ a natural map of vector spaces

$$\psi_l : \mathrm{U}_l(\mathfrak{n}(\underline{n})) \oplus ann_l(v) \to \mathrm{U}_l(\mathfrak{g})$$

defined by

$$\psi_l(x, y) = x + y.$$

Theorem 2.20. *For all $1 \leq l \leq m(\lambda)$ the map ψ_l induce an isomorphism*

$$\mathrm{U}_l(\mathfrak{n}(\underline{n})) \oplus ann_l(v) \cong \mathrm{U}_l(\mathfrak{g})$$

of vector spaces.

Proof. The Theorem follows from Theorem 2.13 and Lemma 2.19. □

Theorem 2.13 is valid over an arbitrary field K. Hence to generalized Theorem 2.20 to an arbitrary field one has to study the generators of the ideal $ann(v)$ over the finite field $\mathbf{F}_p$ with p elements where p is any prime.

Corollary 2.21. *Let B be a basis for E compatible with the flag $E_\bullet(\underline{n})$. Let*

$$B(\underline{n}) = \{x_1, .., x_D\}$$

be the associated basis for $\mathfrak{n}(\underline{n})$ as constructed in 2.2.1. The following holds for all $1 \leq l \leq m(\lambda)$: The set

$$\mathcal{B}(l, \underline{n}, B) = \{x_1^{v_1} \cdots x_D^{v_D}(v) : 0 \leq \sum_i v_i \leq l\} \tag{2.21.1}$$

is a basis for $\mathrm{U}_l(\mathfrak{g})v$ *as vector space.*

$$dim_k(\mathrm{U}_l(\mathfrak{g})v) = \binom{D+l}{D}. \tag{2.21.2}$$

Proof. From Theorem 2.20 it follows there is an isomorphism

$$\mathrm{U}_l(\mathfrak{n}(\underline{n})) \otimes L_v \cong \mathrm{U}_l(\mathfrak{g})v$$

of vector spaces. From this and the Poincare-Birkhoff-Witt Theorem claim 2.21.1 follows. Also

$$dim(\mathrm{U}_l(\mathfrak{n}(\underline{n}))) = \binom{D+l}{D} = dim(\mathrm{U}_l(\mathfrak{g})v)$$

hence claim 2.21.2 follows. The Corollary is proved. □

Example 2.22. *The case* $\mathfrak{sl}(2,K)$*.*

Let $V_\lambda = \mathrm{Sym}^l(E)$ where E is the standard $\mathfrak{sl}(2,K)$-module. It follows V_λ is finite dimensional and irreducible for all $l \geq 1$. If E has basis e_1, e_2 it follows V_λ has highest weight vector $v = e_1^l$. We get the following calculation:

$$ann_k(v) = char_k(\rho)$$

if $1 \leq k \leq l-1$.

$$ann_k(v) = K\{y^l, y^{l+1}, \cdots, y^k\} \oplus char_k(\rho)$$

if $k \geq l$. Here $y \in \mathfrak{sl}(2,K)$ is the following matrix:

$$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

There is an isomorphism of vector spaces

$$\mathrm{U}(\mathfrak{sl}(2,K)) \cong K[y] \oplus char(\rho)$$

where $K[y] \cong \mathrm{U}(\mathfrak{n}_-)$ and $\mathfrak{n}_-$ is the abelian Lie algebra generated by the element y in $\mathfrak{sl}(2,K)$.

Example 2.23. *Symmetric powers of the standard* $\mathrm{SL}(E)$*-module.*

Let $E = K^n$ with basis $e_1, .., e_n$ and let $V_\lambda = \mathrm{Sym}^l(E)$. It follows the vector $v = e_1^l$ is a highest weight vector for the irreducible $\mathrm{SL}(E)$-module V_λ with highest weight $\lambda = lL_1$. It follows from Corollary 2.21 for all $1 \leq k \leq l$

$$dim_K(\mathrm{U}_k(\mathfrak{g})v) = \binom{n-1+k}{n-1}.$$

We get a filtration of V_λ by P-modules

$$\mathrm{U}_1(\mathfrak{g})v \subseteq \cdots \subseteq \mathrm{U}_l(\mathfrak{g})v \subseteq V_\lambda.$$

We get

$$dim_K(\mathrm{U}_l(\mathfrak{g})v) = \binom{n-1+l}{n-1} = dim_K(V_\lambda)$$

hence we get an equality

$$\mathrm{U}_l(\mathfrak{g})v = V_\lambda$$

of P-modules.

Example 2.24. *The adjoint representation.*

Let

$$ad : \mathfrak{sl}(E) \to \mathrm{End}_K(\mathfrak{sl}(E))$$

be the adjoint representation where $E = K^n$. Since $\mathfrak{sl}(E)$ is simple it follows ad is an irreducible representation with highest weight $\lambda = \omega_1 + \omega_{n-1}$. Hence in the notation from this section we get $l_1 = l_{n-1} = 1$ and $l_2, .., l_{n-2} = 0$. It follows we get a strict inclusion

$$\mathrm{U}_1(\mathfrak{sl}(E))v \subsetneq \mathfrak{sl}(E).$$

An explicit calculation shows there is an equality

$$\mathrm{U}_2(\mathfrak{sl}(E))v = \mathfrak{sl}(E).$$

By Corollary 2.21 we get a semi canonical basis $\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{g})v$ for any irreducible $\mathfrak{g}$-module V_λ in the following sense: Any choice of basis B for E compatible with the flag $E_\bullet(\underline{n})$ determines a direct sum decomposition

$$\mathfrak{g} \cong \mathfrak{n}(\underline{n}) \oplus \mathfrak{p}(\underline{n})$$

and a basis $B(\underline{n})$ for $\mathfrak{n}(\underline{n})$ by the construction 2.2.1. This gives by Corollary 2.21 rise to the basis $\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{g})v$. In the case when there is an equality

$$\mathrm{U}_l(\mathfrak{g}(E))v = V_\lambda$$

we get a semi canonical basis for the irreducible $\mathfrak{g}$-module V_λ.

let V_λ be an irreducible $\mathfrak{g}$-module with highest weight $\lambda = \sum_{i=1}^k l_i \omega_{n_i}$ and highest weight vector v and let $1 \leq l \leq m(\lambda)$.

Definition 2.25. Let B be a basis for E compatible with the flag $E_\bullet(\underline{n})$. Let the corresponding basis $\mathcal{B}(l, \underline{n}, B)$ be the *semi canonical basis* for $\mathrm{U}_l(\mathfrak{g})v$ with respect to B.

Example 2.26. *Standard monomial theory.*

The irreducible $\mathfrak{g}$-module V_λ may by the Borel-Weil-Bott formula be constructed geometrically as the global sections of some invertible sheaf on a flag scheme:

$$V_\lambda \cong \mathrm{H}^0(\mathbb{F}, \mathcal{L}(\underline{l})).$$

There exist a natural basis for the cohomology group $\mathrm{H}^0(\mathbb{F}, \mathcal{L}(\underline{l}))$ in terms of *standard monomials*. This basis comes from the Plücker embedding

$$i : \mathbb{F} \to \mathbb{P}(\wedge^{n_1} E^*) \times \cdots \times \mathbb{P}(\wedge^{n_k} E^*)$$

of the flag scheme. The basis $\mathcal{B}(l, \underline{n}, B)$ given in 2.25 is defined in terms of the enveloping algebra $\mathrm{U}(\mathfrak{n}(\underline{n}))$ but is related to the basis given by standard monomials via the inclusion

$$\mathrm{U}_l(\mathfrak{g})v \subseteq \mathrm{H}^0(\mathbb{F}, \mathcal{L}(\underline{l}))$$

of vector spaces. It is an unsolved problem to express the basis $\mathcal{B}(l, \underline{n}, B)$ in terms of the standard monomial basis for $\mathrm{H}^0(\mathbb{F}, \mathcal{L}(\underline{l}))$. There is work in progress on this problem (see [9]).

The basis consisting of standard monomials is expressed in terms of monomials in homogeneous coordinates on the projective spaces $\mathbb{P}(\wedge^{n_i} E^*)$. As seen in the construction above the basis $\mathcal{B}(l, \underline{n}, B)$ is simple to describe since it is expressed in

terms of the enveloping algebra $\mathrm{U}(\mathfrak{n}(\underline{n}))$, its canonical filtration $\mathrm{U}_l(\mathfrak{n}(\underline{n}))$ and the basis $B(\underline{n})$ coming from the flag $E_\bullet(\underline{n}) \subseteq E$ determined by the highest weight

$$\lambda = \sum_{i=1}^{k} l_i \omega_{n_i}.$$

It would be interesting to express $\mathcal{B}(l, \underline{n}, B)$ in terms of standard monomials and to study problems on the flag scheme in terms of the basis $\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{g})v$ and the basis $B(\underline{n})$ for the Lie algebra $\mathfrak{n}(\underline{n})$. Much work has been done on standard monomials and relations to the geometry of flag schemes. See [2] for a geometric approach to standard monomial theory.

Example 2.27. *Subquotients of generalized Verma modules.*

The G-module $\mathrm{U}(\mathfrak{g}) \otimes_K L_v$ is the *generalized Verma module* associated to the P-submodule $L_v \subseteq V_\lambda$. It has a canonical filtration

$$\mathrm{U}_l(\mathfrak{g}) \otimes_K L_v \subseteq \mathrm{U}(\mathfrak{g}) \otimes_K L_v$$

by P-modules.

In general one may for any finite dimensional G-module V and any sub P-module $W \subseteq V$ where $P \subseteq G$ is any parabolic subgroup consider the associated generalized Verma module

$$\mathrm{U}(\mathfrak{g}) \otimes_K W.$$

It has a canonical filtration of P-modules given by

$$\mathrm{U}_l(\mathfrak{g}) \otimes_K W \subseteq \mathrm{U}(\mathfrak{g}) \otimes_K W.$$

Let $ann(W) = \{x \in \mathrm{U}(\mathfrak{g}) : x(w) = 0 \text{ for all } w \in W\}$. It follows there is an exact sequence

$$0 \to ann(W) \otimes_K W \to \mathrm{U}(\mathfrak{g}) \otimes_K W \to V \to 0$$

of G-modules, and exact sequences

$$0 \to ann_l(W) \otimes_K W \to \mathrm{U}_l(\mathfrak{g}) \otimes_K W \to \mathrm{U}_l(\mathfrak{g})W \to 0$$

of P-modules for all $l \geq 1$. Here $\mathrm{U}_l(\mathfrak{g})W \subseteq V$ is the P-module spanned by $\mathrm{U}_l(\mathfrak{g})$ and W.

The subquotient $\mathrm{U}_l(\mathfrak{g})W$ is in [11] interpreted in terms of geometric objects on the flag variety G/P. Let $\underline{\mathrm{mod}}^G(\mathcal{O}_{G/P})$ be the category of finite rank locally free $\mathcal{O}_{G/P}$-modules with a G-linearization and $\underline{\mathrm{mod}}(P)$ the category of finite dimensional P-modules. There is an equivalence of categories

$$\underline{\mathrm{mod}}^G(\mathcal{O}_{G/P}) \cong \underline{\mathrm{mod}}(P)$$

and in [11], Corollary 3.11 we prove the existence of an isomorphism

$$\mathrm{U}_l(\mathfrak{g})W \cong \mathcal{P}^l(\mathcal{E})(x)^* \tag{2.27.1}$$

of P-modules where $\mathcal{P}^l(\mathcal{E})$ is the l'th jet bundle of a G-linearized locally free sheaf $\mathcal{E}$ on G/P. In general one wants to solve the following problems:

(2.27.2) Give a natural basis for $\mathrm{U}_l(\mathfrak{g})W$ generalizing 2.21.1.

(2.27.3) Calculate $dim(\mathrm{U}_l(\mathfrak{g})W)$ as function of l generalizing 2.21.2.

(2.27.4) Interpret $\mathrm{U}_l(\mathfrak{g})W$ in terms of G/P generalizing 2.27.1.

There is work in progress on problem 2.27.2, 2.27.3 and 2.27.4 (see [13]).

Example 2.28. *Canonical bases for semi simple algebraic groups.*

Let G be a semi simple linear algebraic group and V_λ a finite dimensional irreducible G-module with highest weight vector $v \in V_\lambda$ and highest weight λ. We seek a solution to the following problem:

(2.28.1) Calculate for all integers $l \geq 1$ a decomposition

$$\mathrm{U}_l(\mathfrak{g}) \cong W_l \oplus ann_l(v)$$

and construct a basis $x_1, .., x_{C(l)}$ for W_l generalizing the construction in Corollary 2.21. We get an exact sequence of P-modules

$$0 \rightarrow ann_l(v) \otimes_K L_v \rightarrow \mathrm{U}_l(\mathfrak{g}) \otimes_K L_v \rightarrow \mathrm{U}_l(\mathfrak{g})v \rightarrow 0 \tag{2.28.2}$$

for all $l \geq 1$. The exact sequence 2.28.2 gives rise to an isomorphism

$$W_l \otimes L_v \cong \mathrm{U}_l(\mathfrak{g})v$$

of vector spaces. Hence a solution of problem 2.28.1 would give an equality

$$dim(\mathrm{U}_l(\mathfrak{g})v) = C(l).$$

It would also show the set

$$B = \{x_1(v), .., x_{C(l)}(v)\}$$

is a basis for $\mathrm{U}_l(\mathfrak{g})v$ for all $l \geq 1$ giving a solution to problem 2.28.1 for any finite dimensional irreducible module on any semi simple linear algebraic group. There is by [3] a complete description of generators of the annihilator ideal $ann(v)$ as a function of the weight λ. Hence the calculation of the decomposition $\mathrm{U}_l(\mathfrak{g}) \cong W_l \oplus ann_l(v)$ can be done using this set of generators. There is work in progress on this problem (see [13]).

A complete solution of problem 2.28.1 would give the following: It would give a calculation of a complement

$$\mathrm{U}(\mathfrak{g}) \cong W_\lambda \oplus ann(v)$$

of the annihilator ideal $ann(v)$ of the highest weight vector $v \in V_\lambda$ and a basis B for W_λ. Such a basis would give rise to a basis for V_λ in terms of the enveloping algebra $\mathrm{U}(\mathfrak{g})$: There is an exact sequence of G-modules

$$0 \rightarrow ann(v) \otimes_K L_v \rightarrow \mathrm{U}(\mathfrak{g}) \otimes_K L_v \rightarrow V_\lambda \rightarrow 0$$

inducing an isomorphism

$$W_\lambda \otimes_K L_v \cong V_\lambda$$

of vector spaces. Any basis $B = \{z_1, .., z_F\}$ for W_λ will give rise to a basis $B(\lambda) = \{z_1(v), .., z_F(v)\}$ for V_λ.

Much work has been devoted to the construction of canonical bases in finite dimensional irreducible G-modules V_λ. In [7] the author constructs a "universal" canonical basis B in a quantized enveloping algebra U associated to a root system. He shows the basis B specialize to a basis for V_λ for all highest weights λ. It would be interesting to compare Lusztig's basis to the semi canonical basis $\mathcal{B}(l, \underline{n}, B)$ in the case when $G = \mathrm{SL}(E)$ and there is an equality

$$\mathrm{U}_l(\mathfrak{g})v = V_\lambda$$

of vector spaces.

One could speculate about the existence of a "universal" filtration in U equipped with a "universal" canonical basis specializing to the canonical filtration $\mathrm{U}_l(\mathfrak{g})v \subseteq$

V_λ and a canonical basis $\mathcal{B}(l, \underline{n}, B)$ for $\mathrm{U}_l(\mathfrak{g})v$ for some basis B for E compatible with $E_\bullet(\underline{n})$, $l \geq 1$ and any highest weight λ (see [13]).

3. Appendix: Irreducible finite dimensional SL(E)-modules

In this section we give an elementary construction of all irreducible finite dimensional SL(E)-modules using multilinear algebra. The classification of all irreducible finite dimensional SL(E)-modules is well known. We include an explicit construction for the following reason: It is needed in the previous section for calculational purposes.

Assume we are given a finite dimensional irreducible SL(E)-module V_λ with highest weight vector v and highest weight λ. We may by the general theory from [3] assume λ satisfy the following: We may choose integers

$$1 \leq n_1 < n_2 < \cdots < n_k \leq n-1$$

and integers $l_1, .., l_k \geq 1$ such that

$$\lambda = \sum_{i=1}^{k} l_i \omega_{n_i}.$$

Let $\underline{l} = \{l_1, .., l_k\}$. Let $d_1 = n_1$, $d_i = n_i - n_{i-1}$ for $1 \leq i \leq k$ and $d_{k+1} = n - n_k$. Let $n_{k+1} = n$. It follows $\sum_{i=1}^{k+1} d_i = n$. Let $\underline{d} = \{d_1, d_2, .., d_{k+1}\}$. It follows $\underline{d}$ is a partition of n. Let

$$E_i = K\{e_1, .., e_{n_i}\} \tag{3.0.3}$$

for $1 \leq i \leq k+1$ and let $\underline{n} = \{n_1, .., n_k\}$. Let $B_i = \{e_1, .., e_{n_i}\}$. It follows we get a flag

$$E_\bullet(\underline{n}) : 0 \neq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_k \subseteq E_{k+1} = E \tag{3.0.4}$$

of subspaces of E of type $\underline{n}$. The basis B is compatible with the flag $E_\bullet(\underline{n})$. Let $W(\underline{l}, \underline{n})$ be the following G-module: Let $W_i = \wedge^{n_i} E$ for $1 \leq i \leq k$. Let

$$W(\underline{l}, \underline{n}) = \mathrm{Sym}^{l_1}(W_1) \otimes \cdots \otimes \mathrm{Sym}^{l_k}(W_k).$$

The module $W(\underline{l}, \underline{n})$ is not irreducible in general. Let $w_i = \wedge^{n_i}(E_i) \subseteq W_i$. Since $dim(E_i) = n_i$ it follows w_i is a line. Let

$$v = w_1^{l_1} \otimes \cdots \otimes w_k^{l_k}.$$

It follows $v \in W(\underline{l}, \underline{n})$ is a line. Let $P(\underline{n}) \subseteq G$ be the subgroup fixing the flag $E_\bullet(\underline{n})$. It follows

$$g(E_i) \subseteq E_i$$

for all $g \in P(\underline{n})(K)$. Let $\mathfrak{p}(\underline{n}) = Lie(P(\underline{n}))$. It follows an element $x \in \mathfrak{p}(\underline{n})$ is on the following form:

$$x = \begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ 0 & 0 & \cdots & \cdots & * \\ 0 & 0 & 0 & \vdots & A_{k+1} \end{pmatrix}$$

where A_i is a $d_i \times d_i$-matrix with coefficients in K, and $tr(x) = 0$.

Proposition 3.1. *The following holds:*

$$x(v) = (\sum_{i=1}^{k} l_i(tr(A_1) + \cdots + tr(A_i)))v.$$

Proof. The proof is left to the reader as an exercise. □

Let $\lambda = \sum_{i=1}^{k} l_i \omega_{n_i} \in \mathfrak{h}^*$. It follows for all $x \in \mathfrak{h}$

$$x(v) = \lambda(x)v$$

hence the vector v has weight λ.

Let $V_\lambda \subseteq W(\underline{l}, \underline{n})$ be the sub $\mathfrak{g}$-module spanned by the vector v.

Theorem 3.2. *The $\mathfrak{g}$-module V_λ is finite dimensional and irreducible. The vector v is a highest weight vector for V_λ with highest weight λ.*

Proof. The proof follows from Proposition 3.1 and [3], section 7. □

By the general theory any finite dimensional $\mathfrak{g}$-module is a direct sum of irreducible $\mathfrak{g}$-modules, hence Theorem 3.2 gives an explicit construction of all finite dimensional $\mathfrak{g}$-modules in terms of symmetric and exterior products of the standard module E and flags in E. By the general theory it follows we have given an explicit construction of all finite dimensional $\mathrm{SL}(E)$-modules.

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